数理解析研究所講究録
714巻 1990年 130-132

STARLIKENESS OF CERTAIN INTEGRAL

Mamoru Nunokawa (群馬大学教育・ 布川 護)

1. Introduction.

Let A be the class of functions $f(z)$ which are analytic in $E=\{z : |z|<1\}$, with $f(0)=f'(0)-1=0$. A function $f(z)\in A$ is said to be starlike iff

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \qquad \text{in } E.$$

We denote by S^* the subclass of A consisting of functions which are univalently starlike in E.

R. Singh and S. Singh [3] have proved that if $f(z)\in A$ and $\mathrm{Re}f'(z)>0$ in E, then $F(z)\in S^*$, where

$$F(z) = \int_0^z \frac{f(t)}{t}\,dt .$$

In this paper, we will improve the above result.

2. Preliminaries.

In this paper, we need the following lemmata.

LEMMA 1. Let $p(z)$ be analytic in E, $p(0)=1$ and suppose that

$$\mathrm{Re}(\ p(z) + zp'(z)\ ) > -\frac{\log(4/e)}{(2\log(e/2))} \qquad \text{in } E,$$

where $-(\log(4/e)/(2\log(e/2)) = -0.6294\cdots$.

Then we have

$$\mathrm{Re}p(z) > 0 \qquad \text{in } E.$$

We owe this lemma to [1].

LEMMA 2. Let $p(z)$ be analytic in E, $p(0)=1$ and suppose that

$$\mathrm{Re}(\ p(z) + zp'(z)\ ) > 0 \qquad \text{in } E.$$

Then we have

$$|\arg p(z)| < \alpha^* \frac{\pi}{2} \qquad \text{in } E$$

where

$$1 = \alpha^* + \frac{2}{\pi}\,\mathrm{Tan}^{-1}\,\alpha^*$$

and $$0.6383 < \alpha^* < 0.6384.$$

We owe this lemma to [2, Lemma 3].

LEMMA 3. Let $p(z)$ be analytic, $p(0)=1$ and suppose that

$$\mathrm{Re}(\ p(z) + zp'(z)\ ) > 0 \qquad \text{in } E.$$

If $g(z)$ is analytic in E, $g(0)=1$ and if

$$\mathrm{Re}\ p(z)[zg'(z)+g^2(z)+g(z)] > \frac{\log(4/e)}{6}\left(\ \tan^2 \alpha^* \frac{\pi}{2} - 3\ \right) \qquad \text{in } E,$$

then we have

$$\mathrm{Re}\, g(z) > 0 \qquad \text{in } E.$$

We owe this lemma to [2, Lemma 4].

3. Main theorem.

MAIN THEOREM. Let $f(z) \in A$ and suppose that

(1) $$\mathrm{Re} f'(z) > \frac{\log(4/e)}{6}\left(\ \tan^2 \alpha^* \frac{\pi}{2} - 3\ \right) \qquad \text{in } E,$$

where

$$-0.03518 < \frac{1}{6}(\log(4/e))(\tan^2 \alpha^* \frac{\pi}{2} - 3\ ) < -0.03502\ .$$

Then $F(z) \in S^*$, where

(2) $$F(z) = \int_0^z \frac{f(t)}{t}\, dt\ .$$

Proof. From (2), we have

(3) $$F'(0)=1,\ F'(z)=f(z)/z \text{ and } F''(z)=(zf'(z)-f(z))/z^2\ .$$

Then we have

(4) $$\mathrm{Re}(\ zF''(z) + F'(z)\ ) = \mathrm{Re} f'(z)$$
$$> \frac{\log(4/e)}{6}\left(\ \tan^2 \alpha^* \frac{\pi}{2} - 3\ \right) \qquad \text{in } E.$$

From the assumption (1) and from LEMMA 1, we have

(5) $$\mathrm{Re} F'(z) > 0 \qquad \text{in } E.$$

Let us put

$$p(z) = \frac{F(z)}{z}$$

and

$$g(z) = \frac{zF'(z)}{F(z)}\ .$$

Since $p(0)=1$ and

$$\operatorname{Re}(\, zp'(z)+p(z) \,) = \operatorname{Re}F'(z) > 0 \qquad \text{in } E,$$

by LEMMA 2, we have

$$|\arg p(z)| < \alpha^{*}\frac{\pi}{2} \qquad \text{in } E.$$

On the other hand, by an easy calculation, and from (3) and (5), we have

$$\operatorname{Re}\, p(z)[\, zg'(z) + g^{2}(z) + g(z) \,]$$
$$= \operatorname{Re}[\, zF''(z) + 2F'(z) \,] = \operatorname{Re}[\, f'(z) + \frac{f(z)}{z} \,]$$
$$> \operatorname{Re}f'(z) > \frac{1}{6}(\, \tan^{2}\alpha^{*}\frac{\pi}{2} - 3 \,)(\log(4/e)) \quad \text{in } E.$$

Therefore, from LEMMA 3, we have

$$\operatorname{Re}g(z) > 0 \qquad \text{in } E.$$

This shows that

$$\operatorname{Re}\frac{zF'(z)}{F(z)} > 0 \qquad \text{in } E.$$

This completes our proof.

References

[1] M. Nunokawa, Differential inequalities and Caratheodory functions, Proc. Japan Acad., 65, Ser. A, No. 9(1989). (to appear)

[2] ——————, On starlikeness and convexity of certain integral. submitting.

[3] R. Singh and S. Singh, Starlikeness and convexity of certain integral, Ann. Univ. Mariae Curie-Sklodowska. Lublin, XXXV, 16, Ser. A (1981), 145-148.